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NOTE ON LIGHT QUANTA AND THE ELECTROMAGNETIC FIELD

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Abstract

1. Introduction: light quantum theory and quantum electrodynamics.

2. Extreme light quantum theory; spin of quantum; wave equation for quanta; eigenwerte and solutions of the equation; electrostatic quanta; angular momentum and selection rules; Lorentz covariance; theory of the charge-free field; quanta and electrons.

3. Quantum electromagnetics; reformulation to treat progressive and spherical waves; integrals of momentum and angular momentum; new introduction of light quantum equation; distinction between field and corpuscular theories; zero point energy, negative energies, and electromagnetic mass.

4. Interaction of quanta and charges; critique of light quantum theory.

The quantum theory of the electromagnetic field[1] gives a unitary treatment, on the one hand of the electrostatic fields, and, on the other, of the theory of radiation. The distinction between electrostatic and radiation fields may be preserved in the general theory; for it is possible to put, on this theory, the total energy of a system of field and charges in such a form, that the contributions from the two fields remain distinct. In this total energy there are, in addition to the kinetic energy of the charges, terms of three kinds: (a) the electrostatic interaction energy of the charges with each other and with themselves; (b) the energy of the light quanta, to which must be added an infinite constant, corresponding to a half quantum of energy for each component of the radiation field; (c) the interaction energy of the charges and the radiation field, which gives the magnetic interaction of the charges, the infinite proper magnetic energy of the charges, and those interactions which give rise to the absorption, emission, and scattering of light quanta.

The terms (b) and (c) are quite similar to those which appear in the relativistic extension[2] of Dirac's light quantum theory. There are nevertheless points of difference between the two theories. Some of these differences are altogether trivial; but there are a few which are rather deep-lying. We may cite a few examples of this divergence:

A. In the field theory, the electrostatic field is an integral and inevitable part of the electromagnetic field; only when it is included can one establish the invariance of the scheme under space rotations and Lorentz transformations; but in the Dirac theory there is apparently no place for an electrostatic field, and Lorentz covariance may be established without considering such a

[1] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929), and **59**, 168 (1930). Cited as HPI, and HPII.

[2] e.g., I. Waller, Zeits. f. Physik **61**, 837 (1930).

field; on the other hand the behavior of the scheme under space rotations had not been investigated, and it has therefore not been possible to establish conservation laws for angular momentum, nor to derive from them the selection rules.

B. There is in the Dirac theory no analogue to the infinite zero point energy which, on the field theory, must be added to the energy of the quanta in the radiation field. On the other hand we should expect in a completely and consequently corpuscular theory of light to meet with negative energies for the quanta; there is apparently no analogue to these negative energies in the field theory.

Of course all these points are not unconnected; it will be in part the purpose of this paper to study the connection between them, and to see which of the points of difference arise from an incompleteness of the present theory of quanta, and thus may be dissipated by extending the theory, and which are fundamental and persist. It will turn out that in all the points mentioned under A, the light quantum theory, when properly extended, is in full agreement with the field theory; but that in the points of difference B, there is an irresoluble disparity between the two theories. But before we may profitably make the comparison, we shall have on the one hand to develop a somewhat more complete light quantum theory than that of Dirac; and on the other to make minor formal changes in the Heisenberg-Pauli field theory. It will be easy then to answer the questions which we have put.

2.

We shall begin by developing an extreme light quantum theory, and forget for the time all connection with Maxwell's equations. We shall postpone too a treatment of the interaction of quanta and charges, for this treatment is most readily given after the connection between light quantum theory and quantum electrodynamics has been established. Our present problem is to find the wave equation for the de Broglie waves of the quantum; the theory of the field we shall then obtain by a suitable quantization of the de Broglie amplitudes.

In obtaining the wave equation we may be helped by a consideration of the integrals of angular momentum for light quanta. There are in particular two selection rules for the absorption and emission of radiation by matter, which should follow from the conservation of angular momentum, and the properties of the quanta which appear or disappear. These rules apply strictly to angular momentum measured about a space fixed point, and only with high approximation (see Eq. (9)) to that measured about the center of mass of a heavy atom. The rules assert: I. That the component of atomic angular momentum parallel to the direction of motion of the quantum absorbed or emitted changes during the process by one Bohr unit; the sign of the change is determined by the polarization of the quantum; II. That the total angular momentum of the atom may not remain zero during the process of radiation.

There is a third more qualitative rule, which asserts that the probability that on emission or absorption there be a change of more than one unit in total atomic angular momentum, that this probability is small when the wave length of the light λ is large compared to the dimensions a of the atom, and of the order

$$(a/\lambda)^{2\delta}$$

for a change of atomic angular momentum of $1+\delta$ units. To establish this rule one must know something more of the interaction of quanta and charges beyond the validity of the conservation laws; we shall be able to derive it later.

But these rules already tell us what we need to know: the component of angular momentum of a light quantum parallel to its direction of motion must be plus or minus one Bohr unit, according to the polarization of the quantum; the total angular momentum of a quantum about any point must be an integral number of Bohr units, and may not vanish. If these statements are true, the conservation laws will give us the selection rules.

It should be observed of course that these rules refer to the eigenwerte of the angular momentum. There are many radiation processes in which the components of angular momentum of the atom are not determinate; but if we render them determinate, by the use, say of magnetic fields and circularly polarized light, then the rules will hold. We may also mention that the second and third rules, which refer to total angular momentum, could be derived classically, and that this is in fact historically the origin of the selection rules. But classically the first rule makes trouble; for a strictly plane electromagnetic wave has no component of angular momentum parallel to its vector of propagation. Such a component can not arise from orbital angular momentum; one may, if one chooses, ascribe it to a spin of the quantum. But unlike the electronic spin, the spin of the quantum is a whole, not a half unit; yet it has two, and not three-eigenwerte; zero is not an eigenwert. Further, the total angular momentum of a quantum may not vanish; spin and unit orbital angular momentum may not be antiparallel. These are the facts which must guide us.

Now the second order relativistic equation which we should try first for our wave equation is the equation of the retarded potentials, which might serve for a particle free of forces and of vanishing mass

$$\Box\psi = \{\Delta - \partial_t{}^2\}\psi = 0 \quad \text{with} \quad \partial_t = \frac{1}{c}\frac{\partial}{\partial t}\cdot \tag{1}$$

This equation is in several respects unsatisfactory. Here, just as for the electron, we should want a linear equation, in order to obtain a suitable density-flux vector with vanishing divergence. We want, too, more than one kind of quantum for given vector of propagation, since we must describe somehow the polarization of the quantum. Also the integrals of angular momentum come out wrong. The corresponding operator is

$$\boldsymbol{L} = (h/2\pi i)[\boldsymbol{r} \times \boldsymbol{\nabla}].$$

The eigenwerte of the component $\lambda h/2\pi$ parallel to the vector of propagation are zero; and the eigenwerte of the total angular momentum l, with

$$l(l+1) = (4\pi^2/h^2)L^2 \tag{2}$$

are 0, 1, 2, ...

Now some years ago Jordan suggested that we take account of the polarization of light quanta by using[3] Pauli's spin matrices $\boldsymbol{\sigma}$. With their help we can factor (1), and obtain the two component equation

$$\{(\boldsymbol{\sigma}\cdot\nabla) + \partial_t\}\psi = 0.$$

Here ψ no longer behaves as an invariant under space rotations, but as a spinor of the first rank. We know, however, that it will be impossible to associate such a spinor with any electromagnetic field strength or potential; for the whole of classical electromagnetics involves vectors only. Here the angular momentum operator is

$$L = (h/2\pi i)[\boldsymbol{r} \times \boldsymbol{\nabla}] + \frac{h}{4\pi}\boldsymbol{\sigma}$$

and gives for the eigenwerte

$$\lambda = \pm\frac{1}{2}; \quad l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots,$$

This suggests that we should try a three component theory. We introduce a new

$$\psi = (\psi_1, \psi_2, \psi_3)$$

and in place of the $\boldsymbol{\sigma}_i$ three three-row matrices τ_i which are the components of a vector

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3).$$

We let the τ_i satisfy the commutation laws

$$[\boldsymbol{\tau} \times \boldsymbol{\tau}] = i\boldsymbol{\tau} \tag{3}$$

so that they have the eigenwerte 0, ± 1. We shall have occasion to introduce explicit matrices for the τ_i later. We write now for our wave equation

$$\{(\boldsymbol{\tau}\cdot\boldsymbol{\nabla}) + \partial_t\}\psi = 0. \tag{4}$$

The properties of the solutions of this equation are easy to study. The density—flux vector has the components

$$\tilde{\psi}\psi; \ c\tilde{\psi}\tau_i\psi$$

and on the basis of (4) its divergence vanishes. Thus $c\tau$ plays in some respects the part of the velocity; the eigenwerte of the components are 0, $\pm c$. Further the energy and momentum are given by

$$E = (hc/2\pi i)\partial_t; \ p = (h/2\pi i)\boldsymbol{\nabla}$$

[3] P. Jordan, Zeits. f. Physik. 44, 292 (1927).

so that (4) is just

$$E = (p \cdot v). \tag{5}$$

As in Dirac's equation for the electron, velocity and momentum may have opposite sense, and the energy may be negative. To each value of the momenta, there correspond three solutions of (5), with the eigenwerte

$$E = 0, \ \pm pc.$$

If we look for monochromatic solutions of (4), setting

$$\psi_i = e^{-i\omega t} u_i = e^{-2\pi i \nu t} u_i$$

we find

$$(\boldsymbol{\tau} \cdot \boldsymbol{\nabla}) u = i\omega u. \tag{6}$$

From this we deduce

$$\omega[\Delta + \omega^2] u = 0. \tag{6a}$$

Thus any u constant in time is a solution of (4). Solutions which vary in time must satisfy (1).

We may learn more of these solutions by considering a special case. We look for solutions in the form of a plane wave, so that they depend upon the coordinates only through the common factor

$$e^{ikz}.$$

We may then take τ_3 diagonal, and obtain the three solutions of (6)

$$\begin{array}{llll} \omega = kc; & u_1 = e^{ikz}; & u_2 = 0; & u_3 = 0; \\ \omega = 0; & u_1 = 0; & u_2 = 0^{ikz}; & u_3 = 0; \\ \omega = -kc; & u_1 = 0; & u_2 = 0; & u_3 = e^{ikz}; \end{array} \tag{6b}$$

Now with this choice of τ_3, the u_i transform under a three-dimensional rotation like the components of a spinor of the second rank

$$\begin{cases} u_1 \sim 2^{-1/2} e^{i\beta}(x + iy) \\ u_2 \sim -z \\ u_3 \sim 2^{-1/2} e^{-i\beta}(x - iy). \end{cases}$$

The solution with $\omega = 0$ thus has the vector $\boldsymbol{U}$ of which the u's are the "spinor components" parallel to the vector of propagation; whereas for $\omega \neq 0$, this vector lies in a plane perpendicular to the vector of propagation. This suggests that the solutions with $\omega = 0$ will turn out to have something to do with the static solutions of Maxwell's equations, and that the solutions with $\omega \neq 0$ will be connected with the light quantum solutions.

Note on notation. We shall have, in the following to distinguish between three and four vectors, and their spinor components. In general we shall use capitals for the vector, small letters for the spinor, components; further we shall use latin indices for three-greek indices for four-vectors and spinors, and shall use a raised index for contragredience, and adopt the summation convention. Since we shall not use the spinor components in four dimensions, but rather components which are half spinor and half vector, we give here a table of the notation and components:

Thus

$$
\begin{aligned}
S_k &\cdots \{S_x, S_y, S_z\}\\
s_i &\cdots \{2^{-1/2}(S_x + iS_y),\ -S_z,\ 2^{-1/2}(S_x - iS_y)\}\\
s^i &\cdots \{2^{-1/2}(S_x - iS_y),\ -S_z,\ 2^{-1/2}(S_x + iS_y)\}\\
S_\alpha &\cdots \{S_x, S_y, S_z, S_t\}\\
S^\alpha &\cdots \{S_x, S_y, S_z, -S_t\}\\
s_\mu &\cdots \{2^{-1/2}(S_x + iS_y),\ -S_z,\ 2^{-1/2}(S_x - iS_y), S_t\}\\
s^\mu &\cdots \{2^{-1/2}(S_k - iS_y),\ -S_z,\ 2^{-1/2}(S_x + iS_y),\ -S_t\}.
\end{aligned}
$$

Finally, we shall use ∂_i for the spinor components of the gradient ∇_i.

We may define the angular momentum operator by considering an infinitesmal rotation of coordinates. Thus, if we make such a rotation δ_δ about the z-axis, and if the corresponding change in the u's is δu, then the z-component of the angular momentum L_z is given by

$$\delta u = (2\pi i/h)L_z u\delta_\gamma = [x(\partial/\partial^y) - y(\partial/\partial x) + i\tau_3]u\delta_\gamma.$$

So we find

$$L = (h/2\pi i)[\boldsymbol{r} \times \nabla] + (h/2\pi)^{\mathfrak{t}}. \tag{7}$$

The Eq. (6) is invariant under a space rotation, and the components of $\boldsymbol{L}$ are constants of the motion of the quantum. In particular, for plane wave solutions, the component of $\boldsymbol{L}$ parallel to the vector of propagation commutes with the momentum, and has the eigenwerte

$$
\begin{aligned}
\omega = 0;&\quad \lambda = 0\\
\omega - 0;&\quad \lambda = \pm 1.
\end{aligned}
$$

The total angular momentum defined by (2) also commutes with the energy, and has the eigenwerte

$$l = 0, 1, 2, \cdots.$$

The value 0, cannot, however, occur except when $\omega = 0$. For if $l=0$, then

$$\boldsymbol{\tau} = -i[\boldsymbol{r} \times \nabla].$$

If we put this in (6), we get at once

$$\omega u = -([\boldsymbol{r} \times \nabla]\cdot\nabla)u = 0.$$

Thus we see that (6) gives to true light quanta, with $\omega \neq 0$ the correct integrals of angular momentum. From these, and the conservation laws we may deduce the selection rules I and II.

Now to each vector of propagation there are, as we have seen, three solutions of (6); but if we fix the energy, choosing it for instance positive, only one solution survives; we get only one possible value for λ, and thus only one kind of polarization. If we wish to exclude negative energies, then (6) will not give us all the solutions we need, and we shall have to add another set, the solutions of an equation analogous to (4), but with matrices τ' which satisfy, instead of (3)

$$[\boldsymbol{\tau}' \times \boldsymbol{\tau}'] = -i\boldsymbol{\tau}'. \tag{3'}$$

Then we write

$$[(\boldsymbol{\tau}' \; \Delta) + \partial_t]\psi' = 0. \tag{4'}$$

The angular momentum is now

$$L' = h/2\pi i[r \times \nabla] - h/2\pi\boldsymbol{\tau}. \tag{7'}$$

To each k and $\omega \neq 0$, we now have two solutions of (4) and 4′), with $\lambda = \pm 1$ respectively. We can of course write the two systems (4) and (4′) as one, if we use a six component wave function, and reduced six row matrices, which have the τ's and the τ''s along the main diagonal. In this system of equations we can exclude by an auxiliary condition the solutions for which the energy is negative. We have only to use the improper operator $\sqrt{\Delta}$ which was introduced by Landau and Peierls, and to write[3a]

$$\partial_t\psi = -\sqrt{\Delta}\psi.$$

But when we come to introduce the interaction energy of quanta and charges, this condition will no longer be consistent with the equations of motion for the ψ's, since we shall always have terms in this interaction which give rise to a spontaneous emission of quanta of negative energy. It is not possible in this theory to exclude quanta of negative energy.

It will be convenient for us later to have the solutions of (6) which make l and $L_z = m$, say, diagonal; these may be obtained in polar coordinates, with the polar axis parallel to z. For $\omega \neq 0$, $l \neq 0$, we get two such solutions, with

$$\omega = \pm\, kc$$

each of the form

$$u_j^{\omega l m} = e^{i\mu_j\phi} \sum_{s=0,+1} \overset{(m)}{c_{sj}} P_{l-s}^{(\tau\mu_j)} I_{l-s}. \tag{8}$$

Here $\mu_1 = m+1$, $\mu_2 = m$, $\mu_3 = m-1$; the P's are associated Legendre functions of $\cos\theta$, the I's may be given in terms of Bessel's functions by

$$I_q = (kr)^{-1/2} J_{q+1/2}^{(kr)}$$

and the c's are constants which do not in general vanish, and which take on different values according to whether $\omega = \pm kc$. For the choice of the matrices τ_i given in (10), the c's may be taken real, and to satisfy

$$\overset{(m)}{c_{s1}} = \overset{(-m)}{c_{s3}}\,; \quad \overset{(m)}{c_{s2}} = \overset{(-m)}{c_{s2}}\,; \; s = 0, \; \pm 1.$$

[3a] Landau u. Peierls, Zeits. f. Physik **62**, 188 (1930).

For $l=0, \omega=0$, and for $|\mu_j|=l$, the first term on the right side of (8) does not appear.

We may use (8) to derive the third of our selection rules. We shall show in the next sections that the probability of the emission or absorption by an atom of a quantum in a state r is given essentially by the square of the modulus of the integral

$$B = \int dV u_i{}^r s^i$$

where the vector S is the current density vector of the atom. Now when the wave-length $2\pi/k$ of the quantum is large compared to the dimensions a of the region in which S is appreciable, we may expand the functions I in the integral B about the center of mass of the atom, taken as origin, and stationary. Thus we find that if $l_r=1+\delta$, B is small of the order

$$(ak)^\delta.$$

This makes the probability of absorption or emission of a quantum with $l=1+\delta$, small of the order

$$(ak)^{2\delta}$$

and gives, in conjunction with the conservation laws for angular momentum, the third of our selection rules. By a similar argument we may take account of the recoil of the atom as a whole upon absorption or emission. This recoil will be small when the atom is massive compared to the quantum; it gives rise to a probability for a violation of the selection rules for angular momentum measured about the center of mass of the atom. This probability is of the order

$$(hk/Mc)^2 \tag{9}$$

where M is the mass of the atom. For ordinary light this probability is extremely small.

Although, as we have seen, our fundamental Eq. (4) is covariant under a space rotation, it is not Lorentz covariant. From (6a) and (6b), we see that the reason for this noncovariance is the occurrence of zero eigenwerte for the τ's and for ω. This suggests at once that (4) is a degenerate form of a set of four simultaneous equations for four Ψ's, involving three four-row matrices ρ_i, of which the τ_i are certain three-row submatrices; and that these four row matrices ρ_i must have the eigenwerte ± 1, each twice. A simple investigation of the behavior of (4) under a Lorentz transformation will make this clear.

Let us take for definiteness our τ_i in the form

$$\tau_1 = 2^{-1/2}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & +i \\ 0 & -i & 0 \end{pmatrix}; \quad \tau_2 = 2^{-1/2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix};$$

$$\tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

This gives us for the transformation properites of the Ψ_i under a three-dimensional rotation

$$\psi_1 \sim 2^{-1/2}(x + iy)$$
$$\psi_2 \sim - z$$
$$\psi_3 \sim 2^{-1/2}(x - iy).$$

Now consider a Lorentz rotation in the $z-t$ plane. We should expect Ψ_2 to transform like the z-component of a covariant four-vector; let us call the t-component of this vector Ψ_4. Then we see that in the transformed Eq. (4), new terms appear which give just the four-divergence of Ψ. This suggests that we should so extend our matrices τ_i that the fourth row and column refer to Ψ_4, and that the fourth equation makes the four divergence of Ψ vanish. Thus we may take

$$\rho_1 = 2^{-1/2}\begin{pmatrix} 0 & +i & 0 & -1 \\ -i & 0 & +i & 0 \\ 0 & -i & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}; \quad \rho_2 = 2^{-1/2}\begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +i \\ +i & 0 & -i & 0 \end{pmatrix};$$

$$\rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{11}$$

This gives us $\rho_i^2=1$. The ρ_i each have the eigenwerte ± 1, each twice; they no longer satisfy (3).

The system

$$\{(\boldsymbol{\rho}\cdot\boldsymbol{\nabla}) + \partial_t\}\psi = 0 \tag{12}$$

is now Lorentz invariant. Our Eq. (4)

$$\{(\tau\cdot\boldsymbol{\nabla}) + \partial_t\}\psi = 0$$

follows from (12) if we set

$$\psi_4 = 0. \tag{13}$$

But we obtain from (12) the further condition

$$\partial^i\psi_i = \nabla_i\psi_i = 0. \tag{14}$$

The three divergence of the three vector Ψ must vanish. This condition makes the solutions of (4) for which $\omega=0$, reduce to constants. Here again we see the connection between these solutions and those solutions of Maxwell's equations which give the electrostatic field: in the absence of charges there may be no such field.

The equations (4) and (14) are themselves Lorentz covariant, if we transform the ψ's, not as the spatial components of a four-vector, but as the $4-k$ components of a self dual six vector. This six vector $\Psi_{\mu\nu}$ may be completely determined by giving its $4-k$ components in terms of the ψ's:

$$\Psi_{41} = i/2^{1/2}(\psi_1 + \psi_3); \quad \Psi_{42} = 1/2^{1/2}(\psi_1 - \psi_3); \quad \Psi_{43} = -\, i\psi_2; \quad \Psi_{44} = 0.$$

We can of course use a similar procedure for making (4′) Lorentz covariant.

These equations may all be written very simply in spinor notation. For (12) we get

$$\partial^{s}_{\dot r}\psi^{s\dot t} = 0 \qquad (12')$$

while for the system (4) and (14) we have

$$\partial^{s}_{\dot r}\psi^{\dot r\dot t} = 0. \qquad (12'')$$

For the system corresponding to (4′) then

$$\partial^{\dot s}_{r}\psi^{rt} = 0. \qquad (12'')$$

These last two systems have the same form as Maxwell's equations for the empty field.[3b] We should therefore expect to have to add on the right hand side of these systems essentially the four-vector charge and current density of matter; we should then have the correct interaction between quanta and charges. The precise form of these terms will be given in section 3.

This spinorial form for the wave equations lets us see at once a very grave and inescapable defect of the theory. The equations themselves are covariant; but the Lagrangian from which they and their complex conjugate equations may be deduced is not a scalar density. This has important and disastrous consequences, for it means that the energy density of the quanta is not the 4–4 component of a second rank tensor, nor the momentum density the 4–k component of such a tensor. Further, and equally disastrous, the density and flux of the quanta do not form a four-vector, so that these quantities may surely be given no simple physical meaning. All of these results follow at once from the circumstance that one cannot construct an invariant which shall be linear in

$$\partial^{s}_{\dot r}\psi^{\dot r\dot t}$$

and linear in the complex conjugates of the ψ's:

$$\psi^{lm}$$

nor make a four vector bilinear in $\psi^{\dot r\dot t}$ and ψ^{lm}. This is impossible both for the system (12′) and the systems (12″); and we can see at once that it will be impossible for any system in which the ψ's are spinors of even rank—i.e. world vectors and tensors. If we wish to have a positive definite particle density of the form

$$\bar{\psi}_\mu \psi_\mu$$

[3b] O. Laporte and G. Uhlenbeck, Phys. Rev. **37**, 1380 (1931).

which shall be the four component of a four vector, then our wave functions must necessarily be spinors of odd rank. This is the essential ground for the impossibility of a completely satisfactory light quantum theory.

The fact that the energy density of the light quanta cannot be written in a proper form means that the conservation laws for a system of quanta and charges cannot be made covariant. We shall see another expression of this difficulty when we try to write down the interaction energy of the quanta and charges. And this difficulty will persist as long as we want to write our energy density in the form

$$\bar{\psi}\partial_t\psi.$$

If we give up this condition, which is essential to a truly corpuscular theory, we see at once that

$$\psi^{rt}\psi^{lm}$$

gives us a satisfactory energy momentum tensor; this leads then to a system completely equivalent to classical electrodynamics, and the equations (12″) become then just Maxwell's equations for the empty field. By a suitable quantization the classical theory may of course be made to give a theory of the light quantum field. But as already pointed out by Landau and Peierls, it is then no longer possible to define a positive definite light quantum density, nor to treat the quanta as particles.

As long then as we are concerned only with the empty field, our equations will be satisfactory. We have only in some coordinate system to set up (4), and to retain those solutions for which ω does not vanish. This procedure will be Lorentz' invariant, and will give the correct eigenwerete for the dynamical integrals of the quanta. We cannot however extend our theory without great arbitrariness to the case where charges are present, and interact with the quanta. To do this we have first to study the connection between our wave functions Ψ on the one hand, and the electromagnetic potentials and their treatment in the quantum mechanics on the other. It is this inability to find without the help of classical electromagnetics the correct form for the interaction between quanta and charges that makes us believe that the light quantum theory is not in the end very fundamental, and that the analogy between light quanta and electrons, however attractive from the formal point of view, is not very deep lying. But before we pass to the electrodynamical theory, we may say a few words of this analogy.

If we wish to make the transition from the theory of a single quantum to the theory of a field, in which many quanta satisfy the Einstein-Bose statistics, we have only to write down the correct commutation laws for the wave amplitudes Ψ_α.

$$[\tilde{\psi}_\mu(P),\ \psi_\nu(P')] = (h/2\pi i)\delta_{\mu\nu}\delta(P - P');\ [\tilde{\psi}_\mu(P)_1\tilde{\psi}_\nu(P')] = [\psi_\mu(P),\ \psi_\nu(P')] = 0.$$

The theory of such a system is very simple, and very closely analogous to the field theory for Dirac electrons. The Lagrangian is

$$L = -(hi/2\pi i)\bar{\psi}\{(\boldsymbol{\varrho}\cdot\boldsymbol{\nabla}) + \partial_t\}\psi. \tag{16}$$

Setting $\Psi_4=0$ gives us the supplementary condition (14). and the new Lagrangian

$$L = -(hc/2\pi i)\bar{\psi}\{(\boldsymbol{\tau}\cdot\nabla) + \partial_t\}\psi. \tag{16a}$$

We may list here some of the principal operators Ω which gives us the dynamical integrals of the field, together with the eigenwerte which we find for the integrals of these densities: $\bar{\Omega}=\int dV\bar{\psi}\Omega\psi$

	Operator Ω.	Eigenwerte of $\bar{\Omega}$.
Density	1	$0, 1, 2 \cdots$
Flux	$c\tau$	$c(0, \pm 1, \pm 2 \cdots)$
Energy	$(hc/2\pi i(\boldsymbol{\tau}\cdot\nabla)$	$h\nu_r(0, \pm 1, \pm 2, \cdots)$
Momenta	$(h/2\pi i)\nabla$	$h\nu_r/c(0, \pm 1, \pm 2, \cdots)$
Angular momenta	$(h/2\pi i[\boldsymbol{r}\times\nabla]+h/2\pi\boldsymbol{\tau}$	$h/2\pi(0, \pm 1, \pm 2 \cdots)$

It should be observed that the energy may be positive or negative, and that there is no zero point energy.

Now we shall see in our next section that it is possible, by a somewhat different choice of Lagrangian, to pass from this extreme light quantum theory to quantum electrodynamics, in which the energy in the field is always necessarily positive. We might now ask the question: Is it possible to find a similar trick for the theory of the Dirac electron, so that, perhaps at the expense of an infinite zero point energy, there would be only positive electronic energy levels? It turns out that we can find such a trick, rather simply, and we shall do so in our next section. (See Eqs. (37) and (38).) In this treatment a group of noninteracting electrons has only positive energy levels. If, nevertheless, we do not believe that any method of this kind will be of fruitful application, it is because of two major difficulties which appear. The one difficulty is this: if we apply the method to the study of a single electron in a field of force, we find that the dynamical problem is always degenerate: for each energy there are wave functions which we should associate with the motion of particles of positive and negative charge-mass ratio e/m respectively. Radiative transitions between the corresponding states no longer occur; but we find energies and wave functions which are not found in experience; and in particular Klein's paradox, the abnormal transparency of high potential barriers to slow electrons, still persists. The other fundamental difficulty is that with this method, using the new Lagrangian (38), it is not possible to make the electrons satisfy the exclusion principle. For these reasons we cannot regard this attempt to resolve the difficulty of negative electronic energies as satisfactory. We shall not exploit further the analogy between quanta and electrons.

3.

We turn now to the quantum theory of the electromagnetic field. The Lagrangian function for the empty field is

$$L_0 = 1/8\pi(\boldsymbol{E}^2 - \boldsymbol{H}^2) = 1/16\pi[\nabla_\alpha\Phi_\beta - \nabla_\beta\Phi_\alpha]^2. \tag{17}$$

Here $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field strengths, and Φ the four-vector potential. When charges are present we must add to (17) the term

$$L_i = S^\alpha \Phi_\alpha \tag{18}$$

where S is the four-vector charge and current density. By a variation of the Φ's we may deduce Maxwell's equations; by a variation of the S_α, we may, after the addition of suitable inertial terms, deduce the equations of motion of the charges.

If, as in *HPII*, we simplify the analysis by setting

$$\Phi_4 = 0 \tag{19}$$

then the four Maxwell equation

$$\text{div}\, \boldsymbol{E} + 4\pi\rho = 0 \tag{20}$$

no longer follows from a variation of the Φ_i's in L; we have to add (20) as a supplementary condition on our wave functions, in addition to those which now follow from the variation principle: i.e. our wave functions, F say, must fulfill

$$\{\text{div}\, \boldsymbol{E} + 4\pi\rho\}F = 0. \tag{20a}$$

The momenta canonically conjugate to the Φ_i are

$$\Pi_i = -\frac{1}{4\pi c} E_i. \tag{21}$$

We thus have the commutation laws

$$[\Pi_i(P), \psi_j(P')] = (h/2\pi i)\delta_{ij}\delta(P - P'); \; [\Pi_i(P), \Pi_j(P')] = [\Phi_i(P), \phi_j(P')] = 0. \tag{21a}$$

The Hamiltonian is

$$\overline{H}_0 = \int dV\{2\pi c^2 \Pi_i^2 + 1/8\pi[\nabla \times \Phi]^2\}; \; \overline{H}_i = -\int dV S_k \Phi_k. \tag{22}$$

The components of momentum are

$$J_k = \int dV \Pi_i \Delta_k \phi_i. \tag{23}$$

We shall see later the consequences of this disparity in form of the operators for energy and momentum; for the present it will be enough to remember that in spite of this, and in spite of the convention (19), the scheme is Lorentz invariant, in the sense that the equations fulfilled by all gauge invariant quantities, such as the field strength and energy, are covariant under a Lorentz transformation. The equations for the Φ's themselves are not covariant; but one may, after a Lorentz transformation, return to the original equations by a new choice of gauge (Umeichung).

In nearly all the applications of the theory, it is convenient to introduce normal coordinates for the empty field, and to take the amplitudes of the corresponding oscillations as dynamical variables to replace the Φ's. This may be done in a number of ways. Thus in *HPI* and *HPII*, potentials and field strengths are expanded in trigonometric functions of the coordinates, in such a way that the tangential component of $\boldsymbol{H}$ and the normal component of $\boldsymbol{E}$ vanish at the boundary of a fundamental cube of dimensions L, which is chosen large compared to the region in which there are charges. This is equivalent to enclosing the system in a large box with perfectly reflecting walls; and the orthogonal functions represent plane polarized standing electromagnetic waves. Now there are many problems in which it is convenient to have the electromagnetic waves correspond to definite values, say, of the momentum or the angular momentum. For such a treatment one has to use new boundary conditions, and new orthogonal functions. When we carry through the necessary analysis, we shall be able to see very clearly the points of analogy and the points of disparity between the field theory and the light quantum theory; and this, rather than the convenience of the new schemes in application, is the reason why we shall give in some detail the elaboration of methods substantially equivalent to those already in use.

There are three cases which we shall consider; of these the first may be regarded as more or less preliminary; the orthogonal functions are to represent: (a) plane polarized progressive waves of definite momentum;[4] (b) circularly polarized progressive waves; (c) spherical waves of definite angular momentum. The boundary conditions for these three cases are pretty obvious. In order to have only an enumerable set of oscillations, we shall still consider a finite but large volume V. In (a) and (b) we may take this to be a cube of length L, and may require that potentials and field strengths be periodic in this cube. As long as the dimensions of the cube are very large compared to those of the region to which the charges are confined, we may neglect the image fields of the charges in neighboring cubes, and our formulae will give us results which may be readily interpreted. For (c) we must take our fundamental volume spherical, and may for convenience demand that the potentials vanish on the surface of the sphere. It is in all cases quite easy to see which properties of the field are affected by the boundary conditions, and which are independent of them.

According, for (a), we set

$$\begin{aligned} \Phi_i &= (4\pi c/kV)^{1/2} \sum_k \sum_{l=1}^{3} f_{il}{}^k q_{kl} e^{i(\boldsymbol{k}\cdot\boldsymbol{r})} \\ \Pi_i &= (k/4\pi i V)^{1/2} \sum_k \sum_{l=1}^{3} f_{il}{}^k p_{kl} e^{i(\boldsymbol{k}\cdot\boldsymbol{r})} \end{aligned} \tag{24}$$

where $\boldsymbol{k}=(k_1,\ k_2,\ k_3)$; $k_i=(2\pi/L)n_i$; $n_i=0,\ \pm 1,\ \pm 2,\ \cdots$ and where $f_{il}{}^k$ is given by the square matrix

[4] Cf. for this E. H. Kennard, Phys. Rev. **37**, 458 (1931).

$$
\begin{array}{c|ccc}
i/l \rightarrow & 1 & 2 & 3 \\
\downarrow & & & \\
\hline
1 & \epsilon_2(\epsilon_1^2+\epsilon_2^2)^{-1/2} & \epsilon_1\epsilon_3(\epsilon_1^2+\epsilon_2^2)^{-1/2} & \epsilon_1 \\
2 & -\epsilon_1(\epsilon_1^2+\epsilon_2^2)^{-1/2} & \epsilon_2\epsilon_3(\epsilon_1^2+\epsilon_2^2)^{-1/2} & \epsilon_2 \\
3 & 0 & -(\epsilon_1^2+\epsilon_2^2)^{1/2} & \epsilon_3
\end{array}
\quad \text{with} \quad \epsilon_1^k = |k_i|/k.
$$

In order that potentials and field strengths may have at all points real eigenwerte, we must have

$$q_{-kl} = \bar{q}_{kl}; \quad p_{-kl} = \bar{p}_{kl}. \tag{25}$$

We write now $k>0$ when at least two components of k are not negative, and $k<0$ when at least two components are negative. If $k>0$, then $qk<0$. We use Σ_k' for a summation over all $k>0$. We get then the commutation laws (from (21a))

$$[\bar{p}_{kl}, q_{k'l'}] = [p_{kl}, \bar{q}_{k'l'}] = (h/2\pi i)\delta_{lk'}\delta_{ll'}; \; k, k' > 0$$

$$[\bar{p}_{kl}, p_{k'l'}] = [\bar{q}_{kl}, q_{k'l'}] = [p_{kl}, q_{k'l'}] = [p_{kl}, \bar{q}_{k'l'}] = 0; \; k, k' > 0.$$

For the energy we find

$$\overline{H}_0 = E_e + \sum_k{}' \sum_{l=1}^{2} E_{kl}; \; E_e = \sum_k{}' kc\bar{p}_{k3}p_{k3}; \; E_{kl} = kc\{\bar{p}_{kl}p_{kl} + \bar{p}_{kl}q_{kl}\} \tag{26}$$

and for the momenta

$$\bar{J}_i = \sum_k{}' \sum_{l=1}^{3} J_{i,kl}; \; J_{i,kl} = ik_i\{\bar{p}_{kl}q_{kl} - p_{kl}\bar{q}_{kl}\}. \tag{27}$$

The first term in (26) gives the electrostatic energy, and may be evaluated at once by the use of (20); it gives

$$E_e = \tfrac{1}{2}\int dV \int dV' \frac{\rho\rho'}{|\boldsymbol{r} \cdot \boldsymbol{r}'|}. \tag{28}$$

The second term in (26) gives the proper energy of the light quanta. Since

$$[E_{kl}, J_{i,kl}] = 0$$

energy and momentum may be simultaneously reduced to diagonal form. To do this we make the transformation

$$q_{kl} = -i(h/4\pi)^{1/2}\{N_{1kl}^{1/2}\Gamma_{1kl} - \Gamma_{2kl}^{-1}N_{2kl}^{1/2}\};$$

$$\bar{p}_{kl} = i(h/4\pi)^{1/2}\{\Gamma_{1kl}^{-1}N_{1kl}^{1/2} - N_{2kl}^{1/2}\Gamma_{2kl}\};$$

$$p_{kl} = (h/4\pi)^{1/2}\{N_{1kl}^{1/2}\Gamma_{1kl} + \Gamma_{2kl}^{-1}N_{2kl}^{1/2}\};$$

$$\bar{p}_{kl} = (h/4\pi)^{1/2}\{\Gamma_{1kl}^{-1}N_{1kl}^{1/2} + N_{2kl}^{1/2}\Gamma_{2kl}\};$$

where $N_{ikl}=0, 1, 2 \cdots$, and

$$\Gamma_{ikl}f(N_{jk'l'}) = f(N_{jk'l'} - \delta_{ij}\delta_{kk'}\delta_{ll'})\Gamma_{ikl}$$
$$\overset{-1}{\Gamma}_{1kl}f(N_{jk'l'}) = f(N_{jk'l'} + \delta_{ij}\delta_{kk'}\delta_{ll'})\overset{-1}{\Gamma}_{ikl}.$$

Then we get for energy and momenta

$$\begin{aligned} E_{kl} &= hkc/2\pi(N_{1kl} + N_{2kl} + 1) = h\nu_{kl}\{N_{1kl} + \tfrac{1}{2} + N_{2kl} + \tfrac{1}{2}\} \\ J_{ikl} &= hk_i/2\pi(N_{1kl} - N_{2k}). \end{aligned} \tag{29}$$

If we now write for the light quantum terms in the potentials

$$\phi_i' = (4\pi c/kV)^{1/2} \sum_k{}' \sum_{l=1}^{2} f_{il}{}^k\{q_{kl}e^{i(k.r)} + \bar{q}_{kl}e^{-i(k.r)}\} \tag{30}$$

there are four terms in the light quantum interaction energy $S_i\phi_i'$

$$\begin{aligned} S_i\Phi_i' = -\, i(hc/kV)^{1/2} \sum_k{}' \sum_{l=1}^{2} S_i f_{il}{}^k\{ &(N_{1kl}^{1/2}\Gamma_{1kl} - \overset{-1}{\Gamma}_{2kl}N_{2kl}^{1/2})e^{i(k\cdot r)} \\ &- (\overset{-1}{\Gamma}_{1kl}N_{1kl}^{1/2} - N_{2kl}^{1/2}\Gamma_{2kl})e^{-i(k,r)}\} \end{aligned} \tag{31}$$

which involve the factor $e^{i(kr)}$. For each kind of polarization there is a term which gives the absorption of a quantum of momentum $kh/2\pi$, and one which gives the emission of a quantum of momentum $-kh/2\pi$.

There is a zero point energy, but no zero point momentum. We see formally that this difference arises from the circumstance that the J_i are linear, but H_0 is quadratic, in $\partial/\partial x_\mu$. It is this same circumstance that gives us both positive and negative momenta, but only positive energies. If we were to give up the condition that potentials and field strengths were necessarily real, then the energy would no longer be positive definite; in addition to (25), we should also have solutions of the form

$$q'_{-kl} = -\,\bar{q}'_{kl};\quad p'_{-kl} = -\,\bar{p}'_{kl}; \tag{25a}$$

and our energy would have positive and negative eigenwerte

$$E_{kl} = h\nu_{kl}\{N_{1kl} - N'_{1kl} + N_{2kl} - N'_{2kl}\}. \tag{29a}$$

There would be no zero point energy; but in such a system there would be spontaneous gain in energy by atoms at the expense of the field, as well as spontaneous emission. The light quantum theory gives a situation very much like this.

The interaction terms (31) are precisely those obtained by a direct relativistic expansion of Dirac's light quantum theory; they are the most convenient form for the study of the emission, absorption and scattering of hard light.

To treat (b), circularly polarized light, we may make a preliminary transformation

$$\begin{aligned} \phi_1 &= \bar{\phi}_3 = 2^{-1/2}(\phi_x + i\phi_y);\ \phi_2 = -\Phi_3; \\ \pi^1 &= \bar{\pi}^3 = 2^{-1/2}(\Pi_x - i\Pi_y);\ \pi^2 = -\Pi_3. \end{aligned} \tag{32}$$

The new commutation laws are

$$\begin{aligned} [\pi^i(P), \phi_j(P')] &= (h/2\pi i)\delta_{ij}\delta(P - P') \\ [\pi^i(P), \pi^j(P')] &= [\phi_i(P), \phi_j(P')] = 0. \end{aligned} \tag{33}$$

For the Hamiltonian we get

$$H_0 = 2\pi c^2 \pi^i \pi_i + 1/8\pi\{(\bar{\boldsymbol{\tau}}\cdot\bar{\nabla})\bar{\phi}\cdot(\boldsymbol{\tau}\cdot\nabla)\phi\} \tag{34}$$

where the τ's are given by (10). Consider then the equation

$$(\boldsymbol{\tau}\cdot\nabla)u = i\omega u.$$

This is just our wave equation for quanta. We shall choose as our orthogonal functions the normalized solutions u_i of (4), which depend on the coordinates only through the common factor

$$e^{i(k,r)};\ k_i = 2\pi n_i/L;\ n_i = 0, \pm 1, \pm 2, \cdots.$$

We have studied these solutions, and know that to each k there correspond three solutions, with $\omega = 0, \pm kc$. Here we must normalize the argument of the solutions, which we do by requiring that in everv case

$$u_2 = g e^{i(k.r)}$$

where g is real. Then, demanding as before that potentials and field strengths have real eigenwerte, we make the expansions

$$\begin{aligned} \phi_i &= (4\pi c/k)^{1/2} \sum_k{}' \sum_\omega \{q_{k\omega}u_i^{k\omega} + \bar{q}_{k\omega}u_i^{-k\omega}\} \\ \pi^i &= (4\pi i/k)^{-1/2} \sum_k{}' \sum_\omega \{p_{k\omega}u^{i,k\omega} + \bar{p}_{k\omega}u^{i,-k\omega}\}. \end{aligned} \tag{35}$$

Here, of course,

$$u^1 = u_3,\ u^2 = u_2,\ u^3 = u_1.$$

This gives

$$\begin{aligned} [\bar{p}_{k\omega}, q_{k'\omega'}] &= [p_{k\omega}, \bar{q}_{k'\omega'}] = (h/2\pi i)\delta_{kk'}\delta_{\omega\omega'} \\ [\bar{p}_{k\omega}, \bar{q}_{k'\omega'}] &= [p_{k\omega}, q_{k'\omega'}] = [\bar{p}_{k\omega}, p_{k'\omega'}] = [\bar{q}_{k\omega}, q_{k'\omega'}] = 0 \end{aligned}$$

and, for the Hamiltonian

$$\bar{H}_0 = \sum_k{}' kc \sum_\omega \bar{p}_{k\omega}p_{k\omega} + \sum_k{}' \sum_\omega |\omega|\, \bar{q}_{k\omega}q_{k\omega}.$$

As before, the terms with $\omega=0$ give the electrostatic energy (28). The terms with $\omega\neq 0$ give an energy

$$E_{k\omega} = h\,|\,\omega\,|\,/2\pi(N_{1k\omega} + N_{2k\omega} + 1)$$

and momenta

$$J_{i,k\omega} = hk_i/2\pi(N_{1k\omega} - N_{2k\omega}).$$

For the special case that all N's vanish except when k is parallel to z, the component of angular momentum parallel to z may be made diagonal, and has the eigenwerte

$$(h/2\pi)\left\{ \sum_{k,\omega>0}{}' (N_{1k\omega} - N_{2k\omega}) + \sum_{k,\omega>0} (N_{2k\omega} - N_{1k\omega})\right\}.$$

This leads again to the first selection rule of the preceding section.

The interaction energy (31) takes the form

$$S_i\Phi_i' = i(hc/k)^{1/2} \sum_k{}' \sum_{\omega+0} \{ s^i u_i^{k\omega}(N_{1k\omega}^{1/2}\Gamma_{1k\omega} - \Gamma_{2k\omega}^{-1}N_{2k\omega}^{1/2})$$

$$+ s^i u_i^{-k\omega}(N_{2k\omega}^{1/2}\Gamma_{2k\omega} - \Gamma_{1k\omega}^{-1}N_{1k\omega}^{1/2})\}.$$

Here, as in the light quantum theory, the solutions with $\omega\neq 0$ contribute nothing to energy, momentum or field, as long as there are no charges present.

As another example of the use of (34), we may consider the case (c), using for orthogonal functions the solutions of (6) in polar coordinates which vanish on the surface of a large sphere. We have given such solutions in (8); we normalize their phase by choosing the c's real. Here again the solutions with $\omega=0$ give the electrostatic energy (28). For the light quantum terms in the potential we have

$$\begin{aligned}\phi^{i\prime} = &- i(hc/k)^{1/2} \sum_{\omega,l,m=0} \overset{i}{u}_{\omega l0}(N_{\omega l0}^{1/2}\Gamma_{\omega l0} - \Gamma_{\omega l0}^{-1}N_{\omega l0}^{1/2}) \\ &- i(hc/k)^{1/2} \sum_{\omega,l} \sum_{m>0} \{ \overset{i}{u}_{\omega,l,m}(N_{1\omega lm}^{1/2}\Gamma_{1\omega lm} - \Gamma_{2\omega lm}^{-1}N_{2\omega lm}^{1/2}) \\ &+ \overset{i}{u}_{\omega,l,-m}(N_{2\omega lm}^{1/2}\Gamma_{2\omega lm} - \Gamma_{1\omega lm}^{-1}N_{1\omega lm}^{1/2})\}.\end{aligned}$$

The energy of the light quanta is

$$\overline{H}_0 = \sum_{\omega,l} \sum_{m>0} h\,|\,\omega\,|\,/2\pi(N_{1\omega lm} + N_{2\omega lm} + 1) + \sum_{\omega,l,m=0} h\,|\,\omega\,|\,/2\pi(N_{\omega l0} + \tfrac{1}{2}).$$

For the z-component of angular momentum we get

$$L_z = (h/2\pi) \sum_{\omega,l} \sum_{m>0} m(N_{1\omega lm} - N_{2\omega lm}).$$

By considering the possible changes in total angular momentum when one quantum is absorbed or emitted, we may derive the second and third selection rules of the preceding section.

Although in both the field theory and the light quantum theory we have had occasion to use the equations

$$(\boldsymbol{\tau}\nabla)u = i\omega u;\ (\boldsymbol{\tau}\nabla)\phi + \partial_t\phi = 0;$$

there are, even for the empty electromagnetic field, fundamental differences between the two theories. These may be summarized by comparing the Lagrangian functions of the two theories

$$\text{L Q:} \qquad -\ (hc/2\pi i)\ \bar{\phi}[(\boldsymbol{\tau}\cdot\nabla) + \partial_t]\phi \tag{36a}$$

$$\text{Field:} \qquad (1/8\pi)\left\{\,|\,\partial_t\phi\,|^2 - |\,(\boldsymbol{\tau}\cdot\nabla)\phi\,|^2\right\}. \tag{36b}$$

This difference brings with it disparate definitions of the conjugate momenta

$$\text{L Q:} \qquad \pi_i = -\ (h/2\pi i)\bar{\phi}_i$$

$$\text{Field:} \qquad \pi_i = -\ (1/4\pi c)(\partial_t\phi_i)$$

and of the Hamiltonian

$$\text{L Q:} \qquad (hc/2\pi i)\bar{\phi}(\boldsymbol{\tau}\cdot\nabla)\phi = c\pi^i(\boldsymbol{\tau}\cdot\nabla)\phi_i$$

$$\text{Field:} \quad (1/8\pi)\left\{\,|\,\partial_t\phi\,|^2 + |\,(\boldsymbol{\tau}\cdot\nabla)\phi\,|^2\right\} = 2\pi c^2\pi^i\pi_i + (1/8\pi)\,|\,(\boldsymbol{\tau}\cdot\nabla)\phi\,|^2$$

and accounts for the occurrence, in the one case of negative, in the other of infinite positive energies.

The nonoccurrence of a zero point energy in the light quantum theory is of interest in connection with an investigation[5] of Heisenberg on the electromagnetic mass of charges moving with a velocity close to that of light. One must require of a correct electrodynamics that it should give, for the electromagnetic energy-momentum vector of such a charge, a four vector the length of which grows, in comparison with the magnitude of the time component, the energy, negligible as the velocity approaches that of light. Now Heisenberg showed, that with the *HP* field theory, the vector did not have these properties, and that the reason for this failure lay in the infinite zero point of the field. In this respect the light quantum theory is more satisfactory. But as we shall see in the next section, when we come to consider the interaction of quanta and charges, the light quantum theory gives not only to the magnetic proper energy, but also to the magnetic interaction energy of charges, a value radically different from that obtained on the field theory, and one hardly to be reconciled with experiment.

For the treatment of the Dirac electron outlined at the end of the last section, one has only to make in the Tetrode Lagrangian for the electron

$$(hi/2\pi i)\bar{\psi}[\partial_t - (\boldsymbol{\alpha}\cdot\nabla) + i\gamma\alpha_0]\psi \qquad (\gamma = 2\pi mc/h) \tag{37}$$

[5] W. Heisenberg, Zeits. f. Physik **65**, 4 (1930).

a change analogous to that from (36a) to (36b)

$$|\partial_t\psi|^2 - |[(\alpha\cdot\nabla) + i\gamma\alpha_0]\psi|^2. \tag{38}$$

Here one has no condition on the reality of the ψ's , and one obtains two complete systems of terms, corresponding to (25) and (25a) respectively. In each system the motion of a particle with positive and with negative charge-mass ratio e/m is represented. The particles necessarily satisfy the Einstein-Bose statistics.

The interaction energies of field and charges given in this section have all been derived from the electrokinetic term

$$s^\mu\phi_\mu \tag{18}$$

in the Lagrangian. We have now to find what terms must be added to the Lagrangian of the light quantum theory, to give correctly the electrostatic energy of the field, and the elementary probabilities for radiative processes.

4.

When we develop the field theory by the methods of the last section, the interaction terms (18) of the Lagrangian appear finally in two ways, and give rise to the electrostatic energy and the energy of interaction of light quanta and charges. When we set $\psi_4=0$, the supplementary condition on the wave function involves ρ:

$$\operatorname{div} E + 4\pi\rho = 0. \tag{20}$$

With the help of this condition, the terms in the Hamiltonian which arise from those expansion terms of

$$(1/8\pi)E^2 = (2\pi c^2)\pi_i^2$$

for which the orthogonal functions correspond to $\omega=0$, may be evaluated; they give the electrostatic energy of the field. The terms in

$$S_i\Phi_i$$

which appears directly in the Hamiltonian, may be divided into two groups. The one group, for which the expansion functions correspond to $\omega=0$, may be shown, as a consequence of (20), to contribute nothing to the energy. The other group, for which the expansion functions correspond to $\omega\neq 0$, gives the interaction energy of light quanta and charges, and may always be put in the form

$$i(hc^2/|\omega|)^{1/2}\sum_r(N_2^{1/2}\Gamma_r u_r - \Gamma_r^{-1}N_r^{1/2}u_r).$$

Here the u_r are a complete set of orthogonal functions (e.g. solutions of (6)).

Now we may try to introduce into the Lagrangian of the light quantum theory

$$L = -(hc/2\pi i)\bar{\phi}[(\varrho\cdot\nabla) + \partial_t]\phi$$

interaction terms which will give these same results. These terms must (1) have real eigenwerte; (2) be Lorentz invariant; (3) be linear in S, and not involve the derivatives of S, nor any other quantities referring to the configuration of the charges; (4) be linear in ϕ and $\bar{\phi}$. These conditions very nearly determine the form of the terms. The condition (4) shows that we cannot hope to find for the interaction of charges and quanta a form like that given by the Hamiltonian theory in configuration space for the interaction of two charges.

The conditions give for the most general possible form for the interaction terms

$$S^{\mu}\tilde{\alpha}\bar{\phi}_{\mu} + \bar{S}^{\mu}\tilde{\alpha}\phi_{\mu} \tag{39}$$

where α is some operator not involving the field quantities. It is possible to find a formally satisfactory α; but one has to choose a very quaint operator indeed. The theory is satisfactory in the sense that it leads to the value (28) for the electrostatic energy, and gives the same values as the field theory for the probability of absorption or emission of quanta. This last assertion has of course to be modified, because of the occurrence of auanta of negative energy, for which their is no analogue in the field theory.

This occurrence of quanta of negative energy has an important consequence, in that it alters the value of the magnetic interaction energy of charges. The calculation is altogether analogous to the field theory calculation,[6] except that now half the quanta emitted or absorbed have negative energies; and the effect of these processes gives a contribution to the energy of opposite sign to that of the processes involving positive puanta. As a consequence, one can no longer deduce Breit's equation for the second order magnetic energy;[7] in fact the terms of second order in v/c vanish. This is of course in flagrant contradiction with experiment.

The operator α has to be chosen, to obtain the best agreement with the field theory, in the form

$$\alpha = e^{\pi i/4}(hc)^{1/2}\xi.$$

Here ξ is an operator defined by the relations

$$\xi = \partial_t^{-1/2};\ \xi^2\partial_t = \partial_t\xi^2 = 1;\ \xi\cdot \int f(\omega)e^{-i\omega t}d\omega = e^{\pi i/4}c^{1/2}\int f(\omega)\omega^{-1/2}e^{-i\omega t}d\omega;$$

This choice of α gives at once light quantum interaction terms

$$s^i\phi_i' = -\ i(hc)^{1/2}\sum_z \omega_r^{-1/2}(s^i u_{i,r}N_r^{1/2}\Gamma_r - s\bar{u}_{i,r}\Gamma_r^{-1}N_r^{1/2})$$

of the desired form. Here again the terms from $s^i\varphi_i$ for $\omega=0$ give no contribution to the energy. The supplementary condition introduce by setting $\psi_4=0$ here becomes

$$\bar{\xi}^{-1}\operatorname{div}\phi = 2\pi(hc)^{-1/2}e^{-\pi i/4}\rho;\ \rho^{-1}\operatorname{div}\phi = 2\pi(hi)^{-1/2}e^{\pi i/4}\cdot\rho.$$

[6] J. R. Oppenheimer, Phys. Rev. **35**, 461 (1930).
[7] G. Breit, Phys. Rev. **34**, 553 (1929).

With the help of this and the relation

$$\bar{\phi}\partial_t\phi = -i\xi^{-1}\phi\cdot\bar{\xi}^{-1}\phi$$

we find for the energy in the field

$$\overline{H}_0 = -(hc/2\pi i)\int dV\bar{\phi}\partial_t\phi = \sum_{\omega>0}(h\omega_r/2\pi)N_2 + \tfrac{1}{2}\int dV\int dV'\frac{\rho\cdot\rho'}{|(\mathbf{r}-\mathbf{r}')|}$$

as required.

Against this formulation one may urge the gravest objections. Not only is the form of the interaction energy derived entirely with the help of electromagnetic theory; this form itself is unsatisfactory because of the occurrence of the improper operator ξ. Attempts to avoid the introduction of ξ lead at once away from the corpuscular theory to an alternative formulation of the field theory in terms of the complex vector $E+iH$. But to these formal objections to the theory of light quanta one must add that the occurrence of negative energies is in complete discord with experience, both in its direct consequences, which would give an apparent spontaneous absorption of light by matter, and in the value which it gives for the magnetic interaction energy of charges. And we must recall too the difficulties which arose in section 2, where we found it impossible to construct a scalar Lagrangian, a four vector density and flux, or an energy momentum tensor for the light quantum field. For all these reasons the theory developed in this paper is unsatisfactory. It is not impossible that when the occurrence of negative energies for electrons is fully understood, the theory of light quanta will be applicable.